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Multiplicity of positive solutions for a critical quasilinear elliptic system with concave and convex nonlinearities

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ABSTRACT

In this paper, by using the Lusternik–Schnirelmann category, we obtain a multiplicity result for a quasilinear elliptic system with both concave and convex nonlinearities and critical growth terms in bounded domains.

1. Introduction

In this paper, we are concerned with the multiplicity of positive solutions of the following p-Laplacian elliptic system:

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta_p v = \mu |v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geq 3$, $\lambda, \mu > 0$ are parameters and $\alpha, \beta > 1$ satisfying $\alpha + \beta = p^*$ ($p^* := \frac{pN}{N-p}$, $p < N$, denotes the critical Sobolev exponent). We assume that $1 < q < p$. Recently, Hsu in [1] has proved the existence of at least two positive solutions of problem (1) if the pair of the parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. Our purpose here is to reley the number of positive solutions of problem (1) to the topology of Ω. The main result is the following.

Theorem 1. *Assume that $N > p^2$ and $p^* - \frac{N}{N-p} \leq q < p$. Then, there exists $\Lambda_* > 0$ such that for each $\lambda, \mu \in (0, \Lambda_*)$, problem (1) has at least $cat(\Omega) + 1$ distinct positive solutions.*

When $q \geq p$, employing the Lusternik–Schnirelmann category, it was shown in [2] that if $N \geq p^2$ and $2 \leq p \leq q \leq p^*$, then (1) has at least $cat(\Omega)$ distinct solutions for $\lambda, \mu > 0$ small enough. For more similar results, we refer the reader to [3–5]. When $q < p$, in striking contrast, very few is known about the eventual role of the topology of the domain on the multiplicity question of system (1). To establish our main result we follow, as in [3,2], a classical approach and borrow some

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techniques of [6] and arguments developed in [7]. This paper is organized as follows. In Section 2, we fix some notations and give some preliminary results and known facts. In Section 3, we show some technical lemmas which enable us to construct homotopies between Ω and certain sublevel set of the energy functional associated to (1). In Section 4, we prove Theorem 1.

2. Some notations and preliminaries

The space X designates the Sobolev space $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ equipped by its usual norm

$$\|(u, v)\| := \left(\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx\right)^{\frac{1}{p}}.$$

Solutions to problem (1) will be obtained as critical points of the corresponding energy functional

$$I_{\lambda,\mu}(u, v) := \frac{1}{p}\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx - \frac{1}{q}K_{\lambda,\mu}(u^+, v^+) - \frac{2}{\alpha+\beta}R(u^+, v^+), \tag{2}$$

where

$$K_{\lambda,\mu}(u, v) := \int_\Omega (\lambda|u|^q + \mu|v|^q)dx$$

and

$$R(u, v) := \int_\Omega |u|^\alpha |v|^\beta dx.$$

$N_{\lambda,\mu}$ denotes the Nehari manifold related to $I_{\lambda,\mu}$, given by

$$N_{\lambda,\mu} := \{(u, v) \in X \setminus \{0\} : I'_{\lambda,\mu}(u, v)(u, v) = 0\}.$$

We consider the modified functional $\widetilde{I}_{\lambda,\mu}$ defined on $\mathbb{R} \times X$ by

$$\widetilde{I}_{\lambda,\mu}(t, u, v) := I_{\lambda,\mu}(tu, tv).$$

In the sequel, we shall use Λ_* to denote different small parameters.

Lemma 2. *There exists $\Lambda_* > 0$ such that if $\lambda, \mu \in (0, \Lambda_*)$ and for every $(u, v) \in X \setminus \{0\}$ with $(u^+, v^+) \neq (0, 0)$, the real valued function $t \mapsto \partial_t \widetilde{I}_{\lambda,\mu}(t, u, v)$ has a unique positive zero denoted by $t_1(u, v, \lambda, \mu)$ such that*

$$\partial_{tt}\widetilde{I}_{\lambda,\mu}(t_1(u, v, \lambda, \mu), u, v) > 0.$$

Moreover if $R(u^+, v^+) > 0$, the real valued function $t \mapsto \partial_t \widetilde{I}_{\lambda,\mu}(t, u, v)$ has exactly two positive zeros denoted by $t_1(u, v, \lambda, \mu)$ and $t_2(u, v, \lambda, \mu)$ with

$$\partial_{tt}\widetilde{I}_{\lambda,\mu}(t_1(u, v, \lambda, \mu), u, v) > 0 \quad \text{and} \quad \partial_{tt}\widetilde{I}_{\lambda,\mu}(t_2(u, v, \lambda, \mu), u, v) < 0.$$

In particular, we have

$$t_1(u, v, \lambda, \mu) < t(u, v) < t_2(u, v, \lambda, \mu),$$
$$I_{\lambda,\mu}(t_1(u, v, \lambda, \mu)u, t_1(u, v, \lambda, \mu)v) = \min_{0\le t\le t(u,v)} I_{\lambda,\mu}(tu, tv)$$

and

$$I_{\lambda,\mu}(t_2(u, v, \lambda, \mu)u, t_2(u, v, \lambda, \mu)v) = \max_{t\ge 0} I_{\lambda,\mu}(tu, tv),$$

where

$$t(u, v) := \left[\frac{p-q}{2(p^*-q)} \frac{\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx}{R(u^+, v^+)}\right]^{\frac{1}{p^*-p}}.$$

Proof. The proof is almost the same as that in Brown–Wu [8, Lemma 2.6] (or see Tarantello [9, Lemma 1]) and is omitted here. □

Remark 1. We observe from Lemma 2 that we can split $N_{\lambda,\mu}$ into two disjoint parts:

$$N^+_{\lambda,\mu} := \{(t_1(u, v, \lambda, \mu)u, t_1(u, v, \lambda, \mu)v) : (u, v), (u^+, v^+) \in X \setminus \{0\}\},$$
$$N^-_{\lambda,\mu} := \{(t_2(u, v, \lambda, \mu)u, t_2(u, v, \lambda, \mu)v) : (u, v) \in X \setminus \{0\}, R(u^+, v^+) > 0\}.$$

We denote by $c_{\lambda,\mu}$ the following number:

$$c_{\lambda,\mu} := \inf\{I_{\lambda,\mu}(t_2(u, v, \lambda, \mu)u, t_2(u, v, \lambda, \mu)v) : (u, v) \in X, R(u^+, v^+) > 0\}.$$

Lemma 3. *There exists $\Lambda_* > 0$ such that if $\lambda, \mu \in (0, \Lambda_*)$, $I_{\lambda,\mu}$ satisfies the conditions of the mountain pass theorem with the mountain pass level exactly $c_{\lambda,\mu}$.*

Proof. The proof follows immediately from the Sobolev embedding. $\square$

We know also that

$$c_{\lambda,\mu} > 0. \tag{3}$$

Denote

$$S_{\alpha,\beta} := \inf_{(u,v)\in X\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx}{\left(\int_\Omega |u|^\alpha |v|^\beta dx\right)^{\frac{p}{\alpha+\beta}}}.$$

Working as in the proof of [10, Theorem 5], we deduce that

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right) S, \tag{4}$$

where S is the best Sobolev constant, that is

$$S := \inf_{u\in W_0^{1,p}(\Omega)} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\|u\|^p_{L^{p^*}(\mathbb{R}^N)}}.$$

It is well known that S is independent of Ω. Let us consider $\rho_0 > 0$ such that $B(0, 2\rho_0) \subset \Omega$ and define a cut function $\eta \in C_0^\infty(\Omega)$ such that $0 \le \eta \le 1$, $|\nabla \eta| \le C$, $\eta(x) = 1$ for $|x| \le \rho_0$ and $\eta(x) = 0$ for $|x| > 2\rho_0$. For $\varepsilon > 0$, we set

$$u_\varepsilon(x) = \frac{\eta(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}.$$

We know that u_ε satisfies the following estimates:

$$\left(\int_\Omega |u_\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}} = \varepsilon^{-\frac{N-p}{p}} \|U\|^p_{L^{p^*}(\mathbb{R}^N)} + O(\varepsilon),$$
$$\int_\Omega |\nabla u_\varepsilon|^p dx = \varepsilon^{-\frac{N-p}{p}} \|\nabla U\|^p_{L^p(\mathbb{R}^N)} + O(1),$$

where $U(x) := (1 + |x|^{\frac{p}{p-1}})^{-\frac{N-p}{p}} \in W^{1,p}(\mathbb{R}^N)$, which attains S, that is

$$S = \frac{\|\nabla U\|^p_{L^p(\mathbb{R}^N)}}{\|U\|^p_{L^{p^*}(\mathbb{R}^N)}}.$$

By this it follows that

$$\frac{\int_\Omega |\nabla u_\varepsilon|^p dx}{\left(\int_\Omega |u_\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}}} = S + O(\varepsilon^{\frac{N-p}{p}}). \tag{5}$$

Now, we prove the following.

Lemma 4.

$$c_{0,0} = \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}.$$

Proof. Set $u_0 := \sqrt[p]{\alpha}\varepsilon^{\frac{N-p}{p^2}} u_\varepsilon$, $v_0 := \sqrt[p]{\beta}\varepsilon^{\frac{N-p}{p^2}} u_\varepsilon$. By direct computation we obtain

$$\begin{aligned} c_{0,0} \le \sup_{t\ge 0} I_{0,0}(tu_0, tv_0) &= \frac{1}{N2^{\frac{N-p}{p}}} \left(\frac{(\alpha+\beta)\int_\Omega |\nabla u_\varepsilon|^p dx}{\left(\alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} \int_\Omega |u_\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}}}\right)^{\frac{N}{p}} \\ &= \frac{1}{N2^{\frac{N-p}{p}}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right)^{\frac{N}{p}} \left(\frac{\int_\Omega |\nabla u_\varepsilon|^p dx}{\left(\int_\Omega |u_\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}}}\right)^{\frac{N}{p}}. \end{aligned}$$

From (4) and (5) we conclude that

$$c_{0,0} \leq \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}. \tag{6}$$

On the other hand, using the mountain pass theorem, we obtain a Palais–Smale sequence $\{(u_n, v_n)\} \subset X$ for $I_{0,0}$ at level $c_{0,0}$ and, by standard arguments, we show that $\{(u_n, v_n)\}$ is bounded in X. Since

$$\|(u_n^-, v_n^-)\|^p = I'_{0,0}(u_n, v_n)(u_n^-, v_n^-) \to 0,$$

we can assume that $u_n, v_n \geq 0$, so

$$\|(u_n, v_n)\|^p \to l \quad \text{and} \quad 2R(u_n, v_n) \to l.$$

Combining this with the definition of $S_{\alpha,\beta}$, we deduce that

$$\begin{aligned} S_{\alpha,\beta}\left(\frac{l}{2}\right)^{\frac{p}{p^*}} &= S_{\alpha,\beta} \lim_{n\to+\infty} (R(u_n, v_n))^{\frac{p}{p^*}} \\ &\leq \lim_{n\to+\infty} \|(u_n, v_n)\|^p = l, \end{aligned}$$

which imply that

$$l \geq 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}. \tag{7}$$

Now, remarking that $I_{0,0}(u_n, v_n) \to c_{0,0}$ imply that $l = c_{0,0}N$, we conclude from (7) that

$$c_{0,0} \geq \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}. \tag{8}$$

From (6) and (8) we obtain

$$c_{0,0} = \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}. \quad \square$$

Lemma 5. *$I_{\lambda,\mu}$ satisfies the Palais–Smale condition at level c with c satisfying*

$$c \in \left(-\infty, c_\infty := \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}} - K\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)\right),$$

where $K > 0$ is independent of λ and μ.

Proof. See the proof in [1, Lemmas 2.4]. $\square$

Lemma 6. *There exists $\Lambda_* > 0$ such that for $\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_*$, we have*

$$c_{\lambda,\mu} < c_\infty.$$

Proof. The proof is given in Appendix. $\square$

Theorem 7. *There exists $\Lambda_* > 0$ such that if $\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_*$, the functional $I_{\lambda,\mu}$ has a minimizer in $N^-_{\lambda,\mu}$, that is, there exists $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in N^-_{\lambda,\mu}$ satisfying*

$$I_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = c_{\lambda,\mu}.$$

Proof. With Lemma 3 we get a Palais–Smale sequence $\{(u_n, v_n)\} \subset X$ for $I_{\lambda,\mu}$ at level $c_{\lambda,\mu}$. Hence, the proof follows by Lemmas 5 and 6. $\square$

3. Technical lemmas

Here we prove some technical lemmas that we will need to exhibit the necessary homotopies.

Lemma 8. *There exists $\Lambda_* > 0$ such that if (λ_n) and (μ_n) are decreasing sequences in $(0, \Lambda_*)$ converging to 0, then the sequence (c_{λ_n,μ_n}) converges to $c_{0,0}$.*

Proof. First, we note from Lemma 2 and (3) that for all $n \in \mathbb{N}$,

$$0 < c_{\lambda_1,\mu_1} \le c_{\lambda_n,\mu_n} \le c_{0,0}. \tag{9}$$

By Theorem 7, it is possible to consider a sequence $\{(u_n, u_n)\} \subset N^-_{\lambda,\mu}$, $u_n, v_n \ge 0$ such that

$$I_{\lambda_n,\mu_n}(u_n, v_n) = c_{\lambda_n,\mu_n} \quad \text{and} \quad I'_{\lambda_n,\mu_n}(u_n, v_n) = 0. \tag{10}$$

Let (t_n) be a sequence in $\mathbb{R}$ verifying $(t_n u_n, t_n v_n) \in \mathcal{N}_{0,0}$. Taking into account (10), (2) and Lemma 2 we have that

$$\begin{aligned} c_{0,0} &\le I_{0,0}(t_n u_n, t_n v_n) = I_{\lambda_n,\mu_n}(t_n u_n, t_n v_n) + \frac{t_n^q}{q} K_{\lambda_n,\mu_n}(u_n^+, v_n^+) \\ &\le c_{\lambda_n,\mu_n} + \frac{t_n^q}{q} K_{\lambda_n,\mu_n}(u_n^+, v_n^+). \end{aligned} \tag{11}$$

We claim that (t_n) is a bounded sequence. Assume by contradiction that $t_n \to +\infty$. We have from (9), (10) using Sobolev's inequality that

$$c_{0,0} \ge c_{\lambda_n,\mu_n} \ge \frac{1}{N} \|(u_n, v_n)\|^q [\|(u_n, v_n)\|^{p-q} - C(\lambda_n + \mu_n)],$$

where C is independent of n. Since $q < p$, this implies that the sequence $\{(u_n, v_n)\}$ is bounded in X. Then by Sobolev's inequality we deduce that $K_{\lambda_n,\mu_n}(u_n^+, v_n^+) \to 0$. Now, since $(t_n u_n, t_n u_n) \in \mathcal{N}_{0,0}$ then $\|(u_n, v_n)\|^p = 2t_n^{p^*-p} R(u_n^+, v_n^+)$, which imply necessary that $R(u_n^+, v_n^+) \to 0$. We conclude from (10) that $\|(u_n, v_n)\|^p \to 0$ then $c_{\lambda_n,\mu_n} \to 0$, which is a contradiction with (9) and then the claim follows. By (9) and (11), we deduce

$$c_{0,0} \le \liminf_{n\to+\infty} c_{\lambda_n,\mu_n} \le \limsup_{n\to+\infty} c_{\lambda_n,\mu_n} \le c_{0,0},$$

that is, $c_{0,0} = \lim_{n\to+\infty} c_{\lambda_n,\mu_n}$. □

Let us consider

$$\Omega_r^+ := \{x \in \mathbb{R}^N : dist(x, \Omega) < r\},$$
$$\Omega_r^- := \{x \in \Omega : dist(x, \partial\Omega) > r\},$$

where a positive number r will be chosen in such a way that Ω_r^+ and Ω_r^- are homotopically equivalent to Ω. We may assume $B_r := B_r(0) \subset \Omega$. We consider

$$X_r := \{(u, v) \in W_0^{1,p}(B_r) \times W_0^{1,p}(B_r) : u, v \text{ are radial}\}.$$

Let $u \in W_0^{1,p}(B_r)$, we denote with the same symbol u its extension to Ω, with $u = 0$ outside of B_r. Let us consider the functional $I_{\lambda,\mu,B_r} : X_r \to \mathbb{R}$ as

$$I_{\lambda,\mu,B_r}(u, v) := \frac{1}{p} \int_{B_r} (|\nabla u|^p + |\nabla v|^p) dx - \frac{1}{q} K_{\lambda,\mu}(u^+, v^+) - \frac{2}{\alpha + \beta} R(u^+, v^+)$$

and set

$$\widetilde{c}_{\lambda,\mu} := \inf\{I_{\lambda,\mu}(t_2(u, v, \lambda, \mu)u, t_2(u, v, \lambda, \mu)v) : (u, v) \in X_r, R(u^+, v^+) > 0\}.$$

An argument similar to that of Lemma 4 shows the following.

Lemma 9.

$$\widetilde{c}_{0,0} = \frac{2}{N} \left(\frac{S_{\alpha,\beta}}{2} \right)^{\frac{N}{p}}.$$

Lemma 10. *I_{λ,μ,B_r} satisfies the $(\mathrm{PS})_c$ condition with c satisfying*

$$c \in (-\infty, c_\infty).$$

Lemma 11. *There exists $\Lambda_* > 0$ such that if $\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_*$, we have*

$$\widetilde{c}_{\lambda,\mu} < c_\infty.$$

For the proofs of the last two lemmas, we refer the reader to [1, Lemmas 2.4] and Appendix.

The Sobolev embedding permits us to verify that I_{λ,μ,B_r} satisfies the conditions of the mountain pass theorem with the mountain pass level $\widetilde{c}_{\lambda,\mu}$, provided λ and μ are both less than a suitable parameter Λ_*. We conclude from Lemmas 10 and 11 that there exist positive radial functions $u_{\lambda,\mu}, v_{\lambda,\mu} \in N^-_{\lambda,\mu}$ verifying $I_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = \widetilde{c}_{\lambda,\mu}$. Now, let us introduce the following map $\gamma : N^-_{\lambda,\mu} \to \mathbb{R}^N$ given by

$$\gamma(u, v) := \frac{N}{2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}} \int_\Omega x|u|^\alpha |v|^\beta dx.$$

We consider

$$N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}} := \{(u, v) \in N^-_{\lambda,\mu} : I_{\lambda,\mu}(u, v) \le \widetilde{c}_{\lambda,\mu}\}$$

and the map $\delta : \Omega^-_r \to N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}$ given by

$$\delta(y)(x) := \begin{cases} (u_{\lambda,\mu}(x-y), v_{\lambda,\mu}(x-y)) & \text{if } x \in B_r(y), \\ 0 & \text{if } x \notin B_r(y). \end{cases}$$

Taking into account that $u_{\lambda,\mu}$ and $v_{\lambda,\mu}$ are radial, we note that for all $y \in \Omega^-_r$

$$\begin{aligned} \frac{2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}}{N}(\gamma o\delta)(y) &= \int_\Omega x|u_{\lambda,\mu}(x-y)|^\alpha |v_{\lambda,\mu}(x-y)|^\beta dx \\ &= \int_\Omega (z+y)|u_{\lambda,\mu}(z)|^\alpha |v_{\lambda,\mu}(z)|^\beta dz \\ &= \int_\Omega y|u_{\lambda,\mu}(z)|^\alpha |v_{\lambda,\mu}(z)|^\beta dz. \end{aligned}$$

Then $\gamma o\delta$ can be written as follows

$$\gamma o\delta(y) = \pi(\lambda, \mu)y, \tag{12}$$

where

$$\pi(\lambda, \mu) := \frac{N}{2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}} \int_\Omega |u_{\lambda,\mu}(z)|^\alpha |v_{\lambda,\mu}(z)|^\beta dz.$$

Along the way proving Lemma 8 one can check easily the following.

Lemma 12. *We have the following.*

(i) *The map $(\lambda, \mu) \mapsto \widetilde{c}_{\lambda,\mu}$ tends to $\widetilde{c}_{0,0}$, when λ, μ tend to 0.*
(ii) *The map $(\lambda, \mu) \mapsto \pi(\lambda, \mu)$ tends to 1, when λ, μ tend to 0.*

Now we define the map $H_{\lambda,\mu} : [0, 1] \times N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}} \to \mathbb{R}^N$ by

$$H_{\lambda,\mu}(t, u) := \left(t + \frac{1-t}{\pi(\lambda, \mu)}\right)\gamma(u, v).$$

Lemma 13. *There exists $\Lambda_* > 0$ such that for each $\lambda, \mu \in (0, \Lambda_*)$ we have*

$$H_{\lambda,\mu}([0, 1] \times N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}) \subset \Omega^+_r.$$

Proof. We argue by contradiction and suppose that there exist $(t_n) \subset [0, 1]$, $\lambda_n, \mu_n \to 0$ and $(u_n, v_n) \subset N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}$ such that

$$H_{\lambda_n,\mu_n}(t_n, u_n, v_n) \notin \Omega^+_r \quad \text{for all } n \in \mathbb{N}.$$

We can assume that, up to a subsequence, $t_n \to t_0 \in [0, 1]$. By Lemma 12, we have

$$\pi(\lambda_n, \mu_n) \to 1.$$

We have

$$c_{\lambda_n,\mu_n} \le \frac{1}{p}\int_\Omega (|\nabla u_n|^p + |\nabla v_n|^p)dx - \frac{1}{q}K_{\lambda_n,\mu_n}(u_n^+, v_n^+) - \frac{2}{\alpha+\beta}R(u_n^+, v_n^+) \le \widetilde{c}_{\lambda_n,\mu_n}$$

and

$$\int_{\Omega}(|\nabla u_n|^p+|\nabla v_n|^p)dx-K_{\lambda_n,\mu_n}(u_n^+,v_n^+)-2R(u_n^+,v_n^+)=0.$$

As in the proof of Lemma 8, it is easy to verify that the sequence (u_n, v_n) is bounded and by this we obtain

$$c_{\lambda_n,\mu_n}+o(1)\leq\frac{1}{p}\int_{\Omega}(|\nabla u_n|^p+|\nabla v_n|^p)dx-\frac{2}{\alpha+\beta}R(u_n^+,v_n^+)\leq\widetilde{c}_{\lambda_n,\mu_n}+o(1)$$

and

$$\int_{\Omega}(|\nabla u_n|^p+|\nabla v_n|^p)dx-2R(u_n^+,v_n^+)=o(1)$$

as $n\to+\infty$. By Lemmas 4, 8, 9 and 12 it follows that

$$\int_{\Omega}(|\nabla u_n|^p+|\nabla v_n|^p)dx\to 2\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}\quad\text{and}\quad R(u_n^+,v_n^+)\to\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}}.$$

Now, it is easy to see that the sequence $(\widetilde{u}_n,\widetilde{v}_n)$ given by

$$(\widetilde{u}_n,\widetilde{v}_n):=\left(\frac{u_n^+}{R(u_n^+,v_n^+)^{\frac{1}{\alpha+\beta}}},\frac{v_n^+}{R(u_n^+,v_n^+)^{\frac{1}{\alpha+\beta}}}\right)$$

verifies

$$R(\widetilde{u}_n,\widetilde{v}_n)=1\quad\texttt{and}\quad\int_{\Omega}(|\nabla\widetilde{u}_n|^p+|\nabla\widetilde{v}_n|^p)dx\to S_{\alpha,\beta}.$$

By going if necessary to a subsequence, we can assume that

$$\begin{aligned}&(\widetilde{u}_n,\widetilde{v}_n)\to(\widetilde{u},\widetilde{v})\quad\text{a.e. on }\Omega,\\&|\nabla(\widetilde{u}_n-\widetilde{u})|^pdx+|\nabla(\widetilde{v}_n-\widetilde{v})|^pdx\to\mu\quad\text{in }\mathcal{M}(\mathbb{R}^N),\\&|\widetilde{u}_n-\widetilde{u}|^{\alpha}|\widetilde{v}_n-\widetilde{v}|^{\beta}dx\to\nu\quad\text{in }\mathcal{M}(\mathbb{R}^N).\end{aligned}$$

Now, by using similar arguments explored in [6, Lemma 1.40] (see also [11]), we get

$$S_{\alpha,\beta}=\|(\widetilde{u},\widetilde{v})\|^p+\|\mu\|,\qquad 1=R(\widetilde{u},\widetilde{v})+\|\nu\|$$

and

$$\|\nu\|^{\frac{p}{p^*}}\leq S_{\alpha,\beta}^{-1}\|\mu\|.$$

Since

$$(R(\widetilde{u},\widetilde{v}))^{\frac{p}{\alpha+\beta}}\leq S_{\alpha,\beta}^{-1}\|(\widetilde{u},\widetilde{v})\|^p,$$

it is easy to confirm that $R(\widetilde{u},\widetilde{v})$ and $\|\nu\|$ are equal either to 0 or to 1. We see from (4) that $S_{\alpha,\beta}$ is independent of Ω, then a standard argument shows that $S_{\alpha,\beta}$ is never achieved except when $\Omega=\mathbb{R}^N$ (see Remark I.4.7 in [12]), hence it must be $R(\widetilde{u},\widetilde{v})=0$. Again, using similar arguments explored in [6, Lemma 1.40], we deduce that the measure ν is concentrated at a single point y of $\overline{\Omega}$ and so

$$\gamma(u_n,v_n)\to\int_{\Omega}x\,d\nu(x)=y\in\overline{\Omega}\subset\Omega_r^+.$$

Thus

$$H_{\lambda_n,\mu_n}(t_n,u_n)=\left(t_n+\frac{1-t_n}{\alpha(\lambda_n,\mu_n)}\right)\gamma(u_n,v_n)\to y\in\overline{\Omega}\subset\Omega_r^+,$$

which is a contradiction. □

Lemma 14. *There exists $\Lambda_*>0$ such that if $\lambda,\mu\in(0,\Lambda_*)$, we have*

$$cat(N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}})\geq cat(\Omega).$$

Proof. Suppose that $cat(N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}) = n$, this means that n is the least integer such that

$$N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}} = A_1 \cup \cdots \cup A_n,$$

where $A_j, j = 1, \ldots, n$, is closed and contractible in $N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}$, that is, there exists a continuous function $h_j : [0, 1] \times A_j \to N^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}$ such that for all $u, v \in A_j$

$$h_j(0, u) = u \quad \text{and} \quad h_j(1, u) = h_j(1, v).$$

We consider $B_j := \delta^{-1}(A_j), j = 1, \ldots, n$. The sets B_j are closed and

$$\Omega_r^- = B_1 \cup \cdots \cup B_n.$$

Taking into account Lemma 13, let us define the deformation $g_j : [0, 1] \times B_j \to \Omega_r^+$ by setting

$$g_j(t, y) = H_{\lambda,\mu}(t, h_j(t, \delta(y))).$$

This, with (12) gives that for all $y \in B_j$

$$g_j(0, y) = H_{\lambda,\mu}(0, h_j(0, \delta(y))) = \frac{\gamma o \delta(y)}{\pi(\lambda, \mu)} = y$$

and

$$g_j(1, y) = H_{\lambda,\mu}(1, h_j(1, \delta(y))) = \gamma(h_j(1, \delta(y))),$$

which is a fixed quantity. Therefore, we may deduce

$$n \geq cat_{\Omega_r^+}(\Omega_r^-) = cat(\Omega),$$

which completes the proof. □

4. Proof of Theorem 1

We denote by $I_{N_{\lambda,\mu}}$ the restriction of $I_{\lambda,\mu}$ on $N_{\lambda,\mu}$. Before presenting the proof of Theorem 1, we will need here to establish a local compactness condition for the functional $I_{N_{\lambda,\mu}}$ on $N_{\lambda,\mu}$. We set

$$J_{\lambda,\mu}(u, v) := I'_{\lambda,\mu}(u, v)(u, v).$$

Lemma 15. *There exists $\Lambda^* > 0$ such that if $\lambda, \mu \in (0, \Lambda^*)$, then $I_{N_{\lambda,\mu}}$ satisfies the $(\mathrm{PS})_c$ condition with c satisfying*

$$c \in (-\infty, c_\infty).$$

Proof. If $\{(u_n, v_n)\}$ is a Palais–Smale sequence for $I_{N_{\lambda,\mu}}$ at level c, by Willem [6, Proposition 5.12], there exists a sequence $\theta_n \subset \mathbb{R}$ such that $\|I'_{\lambda,\mu}(u_n, v_n) - \theta_n J'_{\lambda,\mu}(u_n, v_n)\|_{X^{-1}} \to 0$. Then

$$I'_{\lambda,\mu}(u_n, v_n) = \theta_n J'_{\lambda,\mu}(u_n, v_n) + o(1). \tag{13}$$

By going if necessary to a subsequence, we may assume that there exists a positive constant K such that

$$1/\|(u_n, v_n)\| \leq K.$$

But on the contrary, up to a subsequence we have $\|(u_n, v_n)\| \to 0$, which completes the proof. Now, we claim that

$$\liminf_{n\to+\infty} |J'_{\lambda,\mu}(u_n, v_n)(u_n, v_n)| > 0,$$

provided that λ, μ are sufficiently small. Indeed, if this is not the case, we get

$$\begin{aligned} o(1) &= (p - q)\|(u_n, v_n)\|^p - 2(p^* - q)R(u_n^+, v_n^+) \\ &= (p - p^*)\|(u_n, v_n)\|^p - (q - p^*)K_{\lambda,\mu}(u_n^+, v_n^+). \end{aligned}$$

By the definition of $S_{\alpha,\beta}$ and the Sobolev embedding theorem it follows that

$$\|(u_n, v_n)\| \geq \left[\frac{p - q}{2(p^* - q)} S_{\alpha,\beta}^{\frac{p^*}{p}}\right]^{\frac{1}{p^*-p}} + o(1)$$

and

$$\|(u_n, v_n)\|^{p-q} \le \frac{p^*-q}{p^*-p} S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}} (\lambda+\mu).$$

We conclude that

$$\lambda+\mu \ge \frac{p^*-p}{p^*-q} S^{\frac{q}{p}} |\Omega|^{\frac{q-p^*}{p^*}} \left[\frac{p-q}{2(p^*-q)} S_{\alpha,\beta}^{\frac{p^*}{p}}\right]^{\frac{p-q}{p^*-p}} + o(1),$$

which is absurd. It follows by (13) that $\theta_n \to 0$ as $n \to +\infty$ and the sequence $\{(u_n, v_n)\}$ is a Palais–Smale sequence for $I_{\lambda,\mu}$. Then the proof follows by Lemma 5. □

Lemma 16. *There exists $\Lambda^* > 0$ such that if $\lambda, \mu \in (0, \Lambda^*)$, then a critical point of $I_{N_{\lambda,\mu}}$ on $N_{\lambda,\mu}$ is a critical point of $I_{\lambda,\mu}$ in X.*

Proof. Arguing as in the proof of Lemma 15. □

Now, we set

$$I_{N_{\lambda,\mu}}^{\widetilde{c}_{\lambda,\mu}} := \{(u, v) \in N_{\lambda,\mu} : I_{\lambda,\mu}(u, v) \le \widetilde{c}_{\lambda,\mu}\}.$$

It is easy to see from Remark 1 and Lemma 14 that

$$cat(I_{N_{\lambda,\mu}}^{\widetilde{c}_{\lambda,\mu}}) \ge cat(\Omega) + 1.$$

Taking into account Lemmas 11 and 15, it follows from the Lusternik–Schnirelmann category (see [6, section 5.3]) that there exist at least $cat(\Omega) + 1$ critical points of $I_{N_{\lambda,\mu}}$ on $N_{\lambda,\mu}$, which are also critical points of $I_{\lambda,\mu}$ in X by Lemma 16.

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Appendix. Proof of Lemma 6

First, we claim that there exist positive constants $C_i (i = 1, 2)$ independent of ε, such that

$$0 < C_1 < t_\varepsilon := t_2(u_0, v_0, \lambda, \mu) < C_2 < \infty,$$

where u_0 and v_0 are given in the proof of Lemma 4. In fact, we have

$$\int_\Omega (|\nabla u_0|^p + |\nabla v_0|^p)dx - 2t_\varepsilon^{p^*-p} R(u_0, v_0) = t_\varepsilon^{q-p} K_{\lambda,\mu}(u_0, v_0).$$

Then, an easy computation implies that

$$2t_\varepsilon^{p^*-p} \le \frac{\int_\Omega (|\nabla u_0|^p + |\nabla v_0|^p)dx}{R(u_0, v_0)} = \frac{(\alpha+\beta)}{\alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}}} \frac{\|\nabla U\|_{L^p(\mathbb{R}^N)}^p}{\|U\|_{L^{p^*}(\mathbb{R}^N)}^{p^*}} + o(\varepsilon^{\frac{N-p}{p}}).$$

We conclude from this that t_ε is bounded above for all $\varepsilon > 0$ small enough. We have by Lemma 2

$$t_\varepsilon \ge t(u_0, v_0),$$

then, we can also assume that t_ε is bounded below. On the other hand, we have

$$\begin{aligned} \int_{B(0,\rho_0)} |u_\varepsilon|^q dx &= \varepsilon^{-\frac{N-p}{p}q} \int_{B(0,\rho_0)} U^q(x\varepsilon^{-\frac{p-1}{p}})dx \\ &\ge \varepsilon^{-\frac{N-p}{p}q + N\frac{p-1}{p}} \int_0^{\rho_0 \varepsilon^{-\frac{p-1}{p}}} U^q(r) r^{N-1} dr \\ &\ge C\varepsilon^{-\frac{N-p}{p}q + N\frac{p-1}{p}} \int_0^{\rho_0 \varepsilon^{-\frac{p-1}{p}}} r^{-q\frac{N-p}{p-1} + N-1} dr. \end{aligned}$$

Now, since $p^* - \frac{N}{N-p} \le q < p$, by a suitable choice of $R_0 > 0$, it follows that

$$\begin{aligned}\int_{B(0,\rho_0)} |u_\varepsilon|^q dx &\ge C\varepsilon^{-\frac{N-p}{p}q+N\frac{p-1}{p}} \int_{R_0}^{\rho_0\varepsilon^{-\frac{p-1}{p}}} r^{-q\frac{N-p}{p-1}+N-1} dr \\ &\ge \begin{cases} C\varepsilon^{-\frac{N-p}{p}q+N\frac{p-1}{p}}, & \text{if } q > p^* - \dfrac{N}{N-p} \\ C\varepsilon^{-\frac{N-p}{p}q+N\frac{p-1}{p}} |\ln\varepsilon|, & \text{if } q = p^* - \dfrac{N}{N-p}, \end{cases}\end{aligned}$$

where C is a positive constant. Then

$$I_{\lambda,\mu}(t_\varepsilon u_0, t_\varepsilon v_0) \le \frac{2}{N}\left(\frac{S_{\alpha,\beta}}{2}\right)^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N-p}{p}}\right) - (\lambda+\mu)\begin{cases} C\varepsilon^{\frac{p-1}{p}(N-q\frac{N-p}{p})}, & \text{if } q > p^* - \dfrac{N}{N-p} \\ C\varepsilon^{\frac{p-1}{p}(N-q\frac{N-p}{p})} |\ln\varepsilon|, & \text{if } q = p^* - \dfrac{N}{N-p}. \end{cases}$$

Since $p^* - \frac{N}{N-p} < q < p$, we can find $\tau > 0$ such that

$$\frac{p-q}{q}\frac{p-1}{p}\left(N - q\frac{N-p}{p}\right) < \tau < \frac{N-p}{p} - \frac{p-1}{p}\left(N - q\frac{N-p}{p}\right). \tag{14}$$

We take

$$\lambda + \mu =: \varepsilon^\tau.$$

Using the fact that $(a+b)^s < a^s + b^s$, if $a, b \ge 0$ and $s \in [0, 1]$, we note that

$$\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \varepsilon^{\tau\frac{p}{p-q}}.$$

We see from (14) that

$$\tau + \frac{p-1}{p}\left(N - q\frac{N-p}{p}\right) < \min\left\{\frac{\tau p}{p-q}, \frac{N-p}{p}\right\}.$$

Then, we can deduce that there exists $\Lambda_* \ge 0$ satisfying for all $\lambda, \mu \in (0, \Lambda_*)$,

$$I_{\lambda,\mu}(t_2(u_0, v_0, \lambda, \mu)u_0, t_2(u_0, v_0, \lambda, \mu)v_0) \le c_\infty$$

and then

$$c_{\lambda,\mu} < c_\infty.$$

If $p^* - \frac{N}{N-p} = q$, we can verify that

$$\frac{p-1}{p}\left(N - q\frac{N-p}{p}\right) < \frac{N-p}{p},$$

then it is easy to see that

$$c_{\lambda,\mu} < c_\infty.$$

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Further reading

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